THE JOHNS HOPKINS UNIVERSITY
Department of Civil Engineering
BALTIMORE, MARYLAND
December, 1953

A DIFFUSION PHENOMENON IN ROTATING AND STRATIFIED FLUIDS

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Technical Report No. 2

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The research reported in this document was supported by The Office of Naval Research under Project NR 082-104, Contract N-onr-248(31).

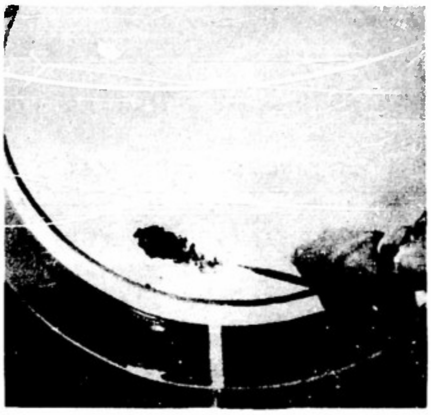
A DIFFUSION PHENOMENON IN ROTATING AND STRATIFIED FLUIDS

Sir Geoffrey Taylor, in one of his first papers on rotating fluids (1921), discovered the following remarkable phenomenon, apparently by accident: If a body of water in solid rotation is disturbed very slightly and a quantity of ink of the same density is dropped into it, the ink, instead of diffusing in a normal manner, gathers into vertical walls.[1] The walls become very thin and very elongated as time goes on, until they can hardly be seen when viewed with a line of sight perpendicular to the axis of rotation. Viewed from above, however, they have the appearance of very thin ink filaments of great total length.

The experiment is reproduced in fig. 1. Just before the ink was inserted, a weak relative motion was generated by a gentle stirring. The Rossby Number, using the radius of the pan, is about 0.03. The angular velocity is 10 rpm. For the purposes of comparison a similar experiment was performed in a non-rotating body of water. The ordinary diffusion process is shown in fig. 2.

Taylor's explanation of this phenomenon was incomplete, and the purpose of this note is to advance a qualitative explanation which seems to serve the purpose. Taylor noted that the relative velocities were so small, compared to the basic rotation, that steady motions had to be two-dimensional, i.e., no vertical motion and no vertical shear. This follows

[1] It is not necessary to disturb the water before putting in the ink. The walls will form without the added disturbance, but they remain so close together that they are more difficult to see in a photograph.



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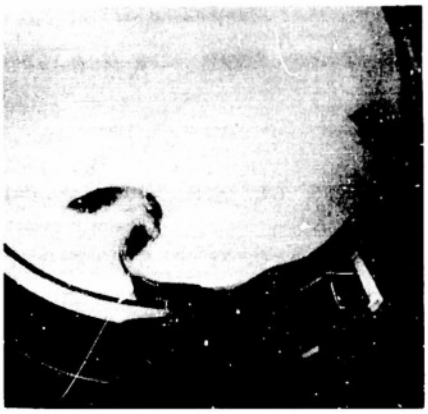

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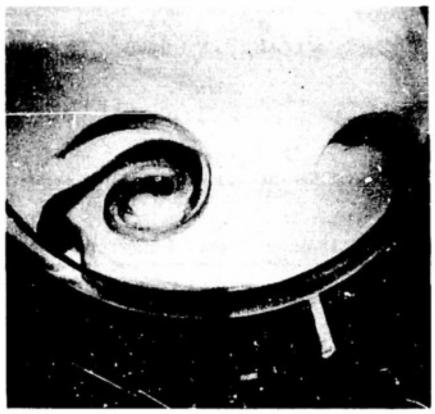

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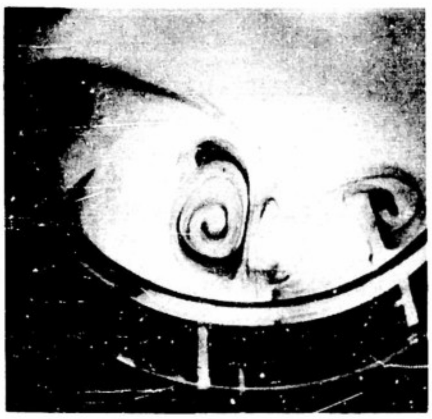

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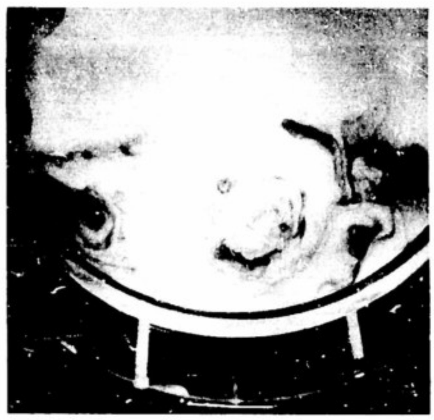
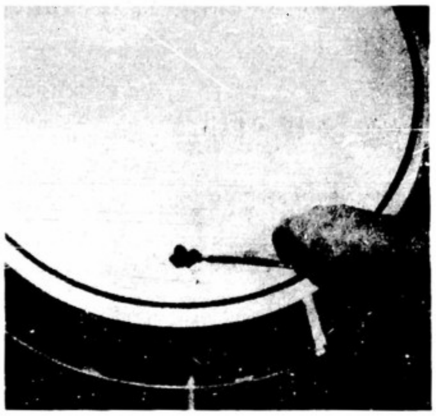


(f) 5 min.

Figure 1. Taylor's ink walls.



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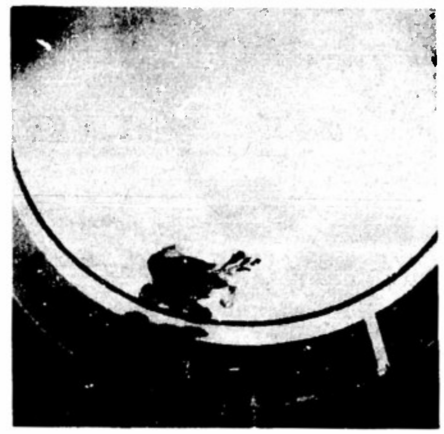

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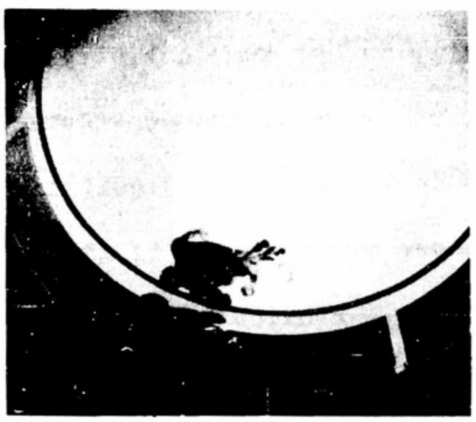
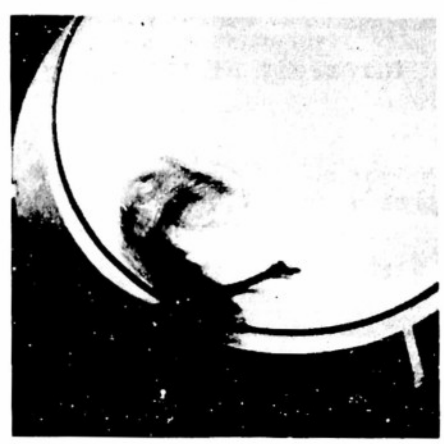


(c) 18 sec.

(d) 30 sec.

(e) 48 sec.



(f) 5 min.

Figure 2. Diffusion in a non-rotating liquid.

from the steady state vorticity equations of a homogeneous frictionless rotating fluid:

$$u\xi_x + v\xi_y + w\xi_z = \xi u_x + \eta u_y + (f+\zeta)u_z$$
$$u\eta_x + v\eta_y + w\eta_z = \xi v_x + \eta v_y + (f+\zeta)v_z$$
$$u\zeta_x + v\zeta_y + w\zeta_z = \xi w_x + \eta w_y + (f+\zeta)w_z$$

where the relative vorticity is (ξ, η, ζ) and the axis of rotation is parallel to the z-axis. As the relative velocities become smaller and smaller all terms involve products of small quantities except fu_z, fv_z, fw_z. If we assume that derivatives of velocity do not remain finite as the velocities become infinitesimal, u_z, v_z, and w_z must become second-order, and ultimately negligible compared to derivatives of velocities in the horizontal direction. Furthermore in our experiment w is zero at the bottom of the vessel. Since its vertical variation is negligible, w is zero everywhere.

The above argument shows that once a vertical wall of ink is formed it will remain as such, undeformed by subsequent motions. For the time intervals of the experiment, molecular diffusion is nearly negligible.

Taylor's explanation is quite correct, but does not explain why the ink gathers originally into walls, nor why the walls become so thin and elongated vertically and horizontally. A rotating fluid possesses vorticity and this is represented by vortex lines parallel to the axis of rotation if there is no relative motion. The number of lines per unit horizontal area is a measure of the intensity of rota-

tion. Neglecting friction, vortex lines are material lines; in all subsequent motions they are composed of the same fluid particles. If we suddenly plunge a foreign body of fluid into a system in solid cyclonic rotation, the vortex lines will be forced apart and as a result the relative circulation on horizontal rings of fluid surrounding the foreign fluid will become negative. Initially, the resulting anticyclonic vortex will not have a balance of pressure gradient and Coriolis force, and the rings will contract. This will force the injected fluid to spread vertically until it reaches the bottom of the vessel and the free surface. The relative motion will now be two-dimensional and the newly-formed ink wall will be stretched out horizontally by local shearing motion.

Such a remarkable type of diffusion process might seem at first glance to be a most unusual occurrence, or, indeed, peculiar to rotating fluids. This is not the case, however; in fact, the above explanation occurred to me while reflecting on the tendency for cigarette smoke in a quiet room to concentrate into sheets of smoke parallel to the floor. A similar phenomenon exists in the atmosphere when smoke is discharged into the air during stable conditions. Using the above approach, the injection of the smoke causes a local separation along the vertical of the surfaces of constant density. The action of gravity, in attempting to restore the horizontality of these surfaces, will compress the smoke vertically and spread it horizontally into sheets. The analogy to the rotating phenomenon is very close.

A more intuitive explanation of the formation of sheets of smoke in a stable atmosphere is that the smoke rises under the action of gravity until it reaches an inversion surface. This prevents any further vertical motion and the smoke spreads horizontally. In order to check the analogy to a rotating system, and to dismiss the necessity for a near-discontinuity in density gradient, implied by the last argument, an experiment was conducted with ink injected into a stable water and salt solution. By careful preparation a vessel was filled with a mixture having a smooth density-height curve (fig. 3). A quantity of ink, with a density equal to the mean density of the fluid in the vessel, was then injected. The subsequent developments are shown in fig. 4. The ink forms a fairly thin sheet within approximately 5 min. After a half-hour or so, the gravitational diffusion effect, tending to spread the ink horizontally and contract it vertically, is reduced to the level of importance of molecular diffusion. Before this occurs, the diffusion process, if it were Fickian, (Sutton, 1953), would involve large horizontal diffusion coefficients and a negative vertical coefficient.

It is of some interest to investigate the time interval required for the formation of the ink walls in a rotating fluid. This may be done crudely by a dimensional argument. Assuming that a spherical volume V of foreign material is injected into an unlimited rotating fluid of the same density, and denoting the decreasing horizontal dimension by d, we have simply

$$d = V^{\frac{1}{3}} \chi(\Omega t).$$

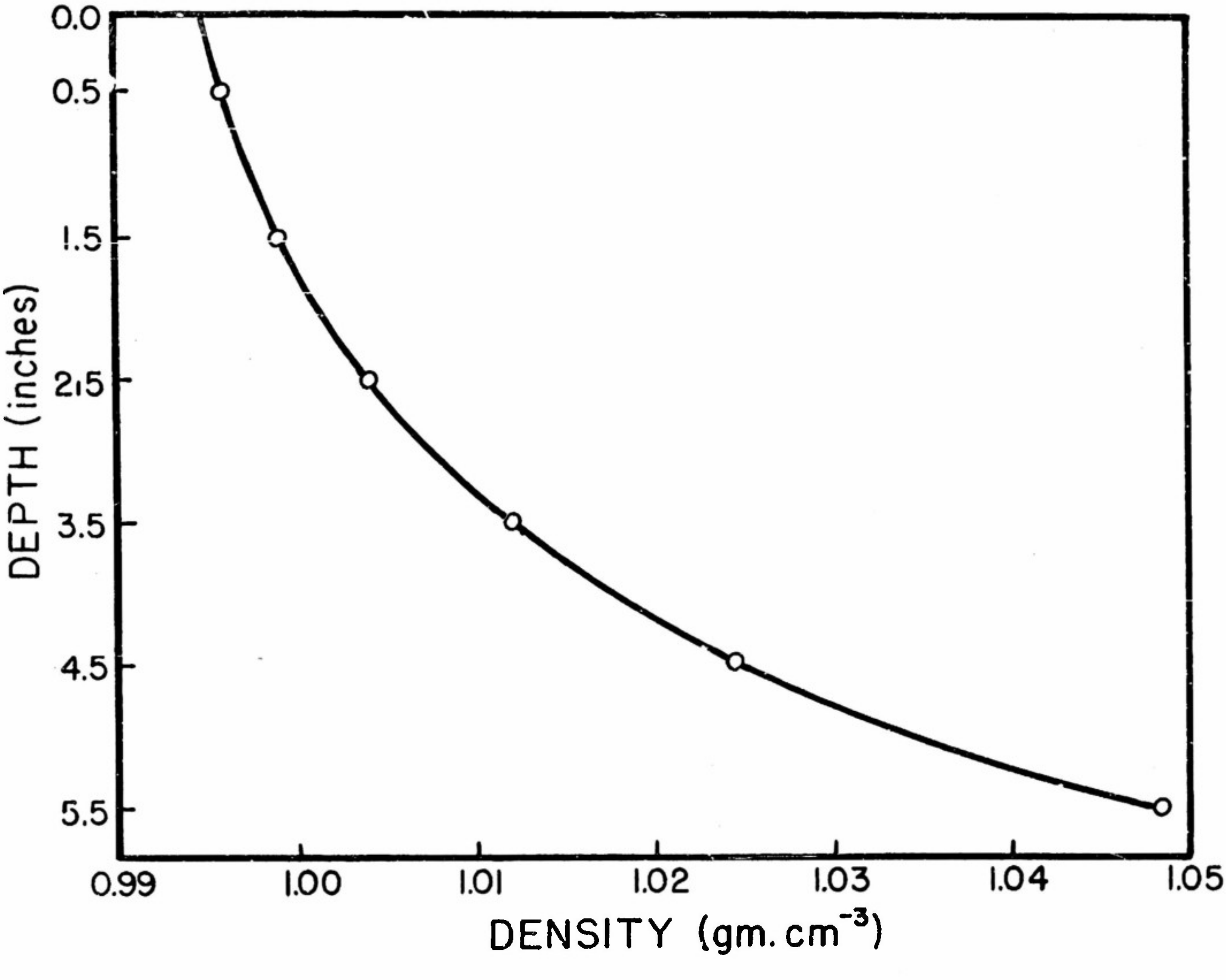
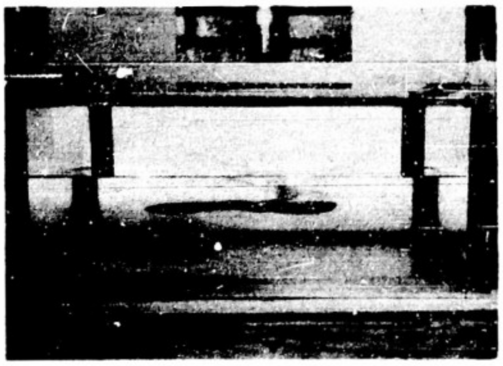
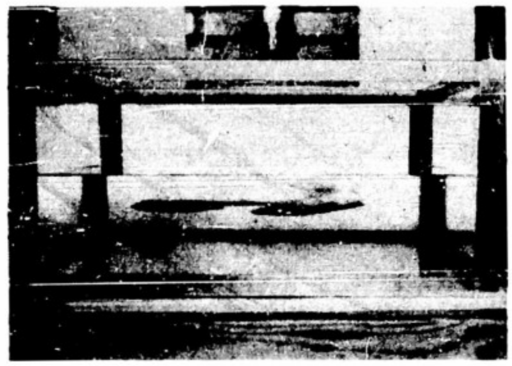


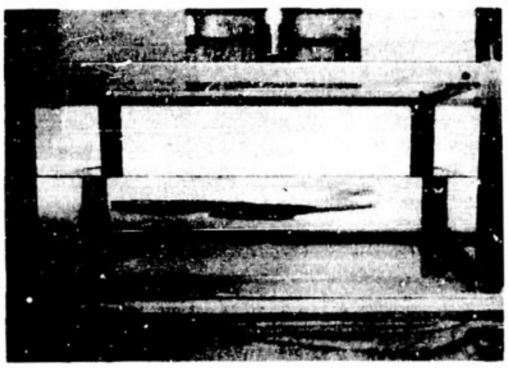
Figure 3. Density-height curve for experiments of fig. 4.



(a) 32 sec.



(b) 51 sec.



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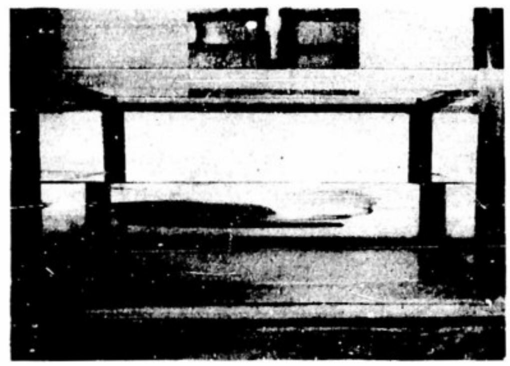
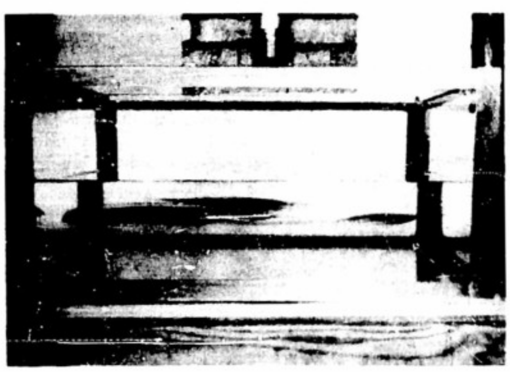


(d) 4 min.



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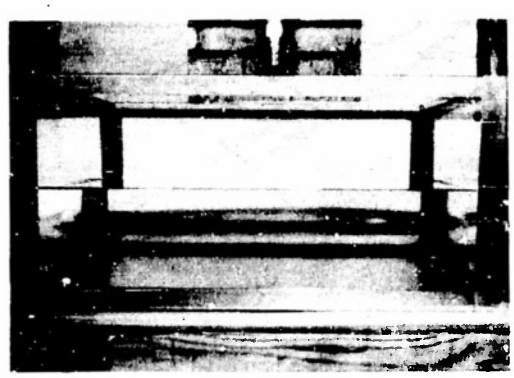

(f) 42 min.

Figure 4. Formation of ink sheets in a stratified fluid.

This assumes, of course, that the volume retains its initial axial symmetry so that it spreads vertically and not horizontally. The experiments[2] indicate that d changes by a factor of 1/10 in perhaps 30 sec at an angular velocity, $\Omega = 1\ sec^{-1}$. Thus a change to $d/V^{\frac{1}{3}}$ of 1/10 is associated with a value of 30 for Ω t. In the atmosphere, where Ω is about $7\times10^{-5} sec^{-1}$, such a change would require $t = 30/7\times10^{-5} sec$ or about 5 days. For any conceivable atmospheric phenomenon of this kind, the effect of eddy diffusivity would be several orders of magnitude greater.

The analogous effect in a stratified fluid would seem to be of more practical importance. A dimensional analysis of this case leads to

$$d = V^{\frac{1}{3}} \phi \left(\frac{V^{\frac{1}{3}} a}{\rho_o} , \frac{g t^2 a}{\rho_o} \right),$$

where d is the diminishing vertical dimension, ρ_o is the uniform density of the foreign matter, a is the density (potential temperature) gradient of the environment, and g is gravity. The first term in the argument of ϕ is the non-dimensional density difference between the ink mass and its environment. If we regard this quantity as important only insofar as it is associated with the buoyancy effect (and therefore only when associated with g), we may neglect this number. In the experiment a/ρ_o was about $3\times10^{-3} cm^{-1}$, and the

[2]The experiment is limited vertically by the bottom of the vessel and the free surface. A full analysis should include this dimension and gravity.

time t for a reduction of d to 1/10 of its original value was about 5 min. If we choose a mass in a stable environment with a potential temperature rise of 5-10C in a hundred meters, the requirement that the second argument of ϕ be equal in the laboratory and the atmosphere yields a value of 1-3 hours for the mass in the atmosphere to form a sheet. This assumes, of course, a complete absence of eddy diffusivity.

This time period seems too long if the above process is responsible for the frequent tendency for smoke to concentrate into sheets in a stable atmosphere. Reports of this phenomenon (Barad, 1951) indicate that it occurs in a matter of minutes. It is possible that the local stability at the height of the stack may be considerably greater than that assumed in the above calculation. It is more likely, however, that the above analysis is too crude. In any event, the vertical eddy diffusivity drops to exceedingly small values if the stability is high. It is, therefore, reasonable to assume that the gravitation effect, tending to condense vertically and spread horizontally, is frequently of the order of magnitude or greater than the vertical eddy diffusivity. This effect, may, moreover, help to explain the findings by Parr (1936) that stability increases horizontal diffusivity.

Acknowledgment

I would like to acknowledge the assistance of Messrs. James F. Haubert and Edwin C. Brockenbrough in conducting the experiments for this work.

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APPENDIX

It seems desirable to present a theoretical development which helps to explain the analogy between rotating and stratified fluids with respect to the diffusion phenomenon described above. We divide the investigation into two parts:

A. Stratified fluid.

According to the description of the experiments the introduction of ink serves initially to distort the density distribution in one case, and the pattern of vortex lines in the other. In this portion of the investigation we will assume a disturbance, ρ', of the density field. The latter is given by $\rho_0\ e^{-\beta z}$, when undisturbed. The coordinates are z, upward, x, horizontal. For simplicity we assume that the disturbance is independent of the y-direction. This does not conform to the situation in the experiment or in the atmosphere but will serve sufficiently well for the present purpose. The exponential density distribution was chosen for simplicity. If we take the origin of the z-axis near the disturbance, βz will be very small in the vicinity of the disturbance and the density gradient will be nearly linear in this region. In fact, we will assign a length L for the characteristic vertical dimension of the disturbance and assume $\beta L \ll 1$. Then, if z_0 is the undisturbed height of a given density surface, $\rho' = \rho_0(e^{-\beta z_0} - e^{-\beta z}) \cong \rho_0 \beta (z - z_0) = o(\rho_0 \beta L)$. From this we infer that $\rho'/\rho_0 \ll 1$, an entirely reasonable result of the assumption, $\beta L \ll 1$.

The vorticity equation is

$$\frac{d\zeta}{dt} + \frac{1}{\rho}\rho_x \frac{dw}{dt} - \frac{1}{\rho}\rho_z \frac{du}{dt} + \frac{1}{\rho}\rho_x g = 0 , \tag{1}$$

$$\zeta = w_x - u_z , \tag{2}$$

where partial derivatives are denoted by subscripts as before. Since the pressure distribution is very closely hydrostatic we may neglect the second term, compared to the last term in equation (1). We assume now that the motion is very slow and neglect the non-linear terms in the velocity and density perturbation. Equation (1) becomes

$$\frac{\partial\zeta}{\partial t} - \frac{\rho_z}{\rho} u_t + \frac{\rho_x}{\rho} g = 0 . \tag{3}$$

By the same assumptions the requirement of incompressibility leads to

$$\rho_t = w \rho_0 \beta . \tag{4}$$

Introducing the stream function by the expressions

$$u = -\psi_z , \quad w = \psi_x , \tag{5}$$

the vorticity equation becomes

$$\nabla^2\psi_{tt} - \beta\psi_{ztt} + g\beta\psi_{xx} = 0 , \quad \nabla^2\psi = \psi_{xx} + \psi_{zz} . \tag{6}$$

The ratio of the second term to the first term is of the order of βL, so that we have, approximately,

$$\nabla^2\psi_{tt} + g\beta\psi_{xx} = 0 . \tag{7}$$

Since initial conditions are given in terms of the disturbed positions of the density surfaces, we may introduce the quantity $\underline{\delta} = z - z_o$. We obtain finally

$$\nabla^2 \delta_{ttt} + g\beta\delta_{xxt} = 0 \, . \tag{8}$$

B. Rotating fluid.

Assuming axial symmetry of the disturbance about the x-axis (axis of rotation), the equations of motion are

$$\frac{dv}{dt} - \frac{w^2}{r} - 2\Omega w = -\frac{\partial \chi}{\partial r} \, , \tag{9}$$

$$\frac{du}{dt} = -\frac{\partial \chi}{\partial x} \, , \tag{10}$$

$$\frac{d}{dt}(\Omega r^2 + wr) = 0 \, , \tag{11}$$

where $\chi = p/\rho$ plus centrifugal and external forces, v is the radial velocity, u is the velocity along the x-axis, w is the velocity tangent to circles about the x-axis, and r is radial distance.

The vorticity equation is

$$r\frac{d}{dt}\left(\frac{\xi}{r}\right) - \frac{2ww_x}{r} - 2\Omega w_x = 0 \, , \tag{12}$$

$$\xi = v_x - u_r \, . \tag{13}$$

Neglecting second-order terms in the velocities, this becomes

$$\xi_t - 2\Omega w_x = 0 \,, \tag{14}$$

and, to the same order of approximation, (11) reduces to

$$w_t + 2\Omega v = 0 \,. \tag{15}$$

Introducing the stream function by the expressions

$$u = -\frac{1}{r}\psi_r \,, \quad v = \frac{1}{r}\psi_x \tag{16}$$

and eliminating w, the vorticity equation is

$$\nabla^2\psi_{tt} + 4\Omega^2\psi_{xx} = 0 \,, \quad \nabla^2\psi = \psi_{xx} + \psi_{rr} - \frac{\psi_r}{r} \,. \tag{17}$$

We now introduce $\delta' = r^2 - r_0^2$, where r_0 is the Lagrangian distance of a vortex line from the x-axis. This expression occurs in the integral of (11):

$$\Omega r^2 + wr = \Omega r_0^2 \,. \tag{18}$$

In terms of δ', (17) becomes

$$\nabla^2\delta'_{ttt} + 4\Omega^2\delta'_{xxt} = 0 \tag{19}$$

It is remarkable that equations (8) and (19), governing the development of an initial disturbance in a stratified and rotating fluid, respectively, are almost identical except for the fact that the Laplacean opera-

tor is in cartesian coordinates in one case and in cylindrical coordinates in the other. The parameters $g\beta$ and $4\Omega^2$ show that the time scales of the two phenomena are in the ratio

$$T_s/T_r = 2\Omega/(g\beta)^{\frac{1}{2}} \tag{20}$$

so that geometrically similar initial disturbances should develop in time periods having this ratio, provided the fluids are effectively infinite or the boundary conditions are geometrically similar. For the experimental conditions of this paper this ratio is about one, whereas the experimental results indicate roughly that the ratio is about 10 (300 sec/30 sec). Possibly this results from the fact that the ink was injected along a line perpendicular to the density surfaces in one case and parallel to the vortex lines in the other.[1] It is physically clear that this would make a considerable difference in the experimental results. The theoretical indication that the time period in the stratified case may be as little as 1/10 of the 1-3 hr period computed from dimensional considerations of the experimental results, is interesting with regard to smoke diffusion in the atmosphere. If smoke issues from a stack at a temperature close to that of the air, the injection would be more or less along a line parallel to the density surface and the vertical flattening may occur in a matter of minutes instead of the 1-3 hr period suggested by the experimental evidence.

[1]We assume here that the different meanings of δ and δ' $(z - z_0,\ r^2 - r_0^2)$, and the two-dimensionality of the stratified portion of the theory are insufficient to explain the order-of-magnitude difference of the time-scale ratios in experiment and theory. A more involved theoretical investigation would be needed to decide this.

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